ON THREE-DIMENSIONAL MAGNETOHYDRODYNAMIC FLOWS

(O PROSTRANSTVENNYKH MAGNITOGIDRODINAMICHESKIKH TECHENIIAKH)

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In [1] we considered plane and axisymmetric magnetohydrodynamic flows of an ideal gas in the presence of a magnetic field parallel to the flow velocity. In that paper, it was shown that two regions of hyperbolic flows exist. In the supersonic hyperbolic region the flow is qualitatively similar to supersonic flow of an ordinary gas. In the subsonic hyperbolic region shock waves extend upstream from the body, so that the flow picture is like that of an ordinary gas flow directed in the opposite direction.

We shall show that within the limits of accuracy of linearized theory this fact also holds for three-dimensional flows, and that extending upstream will be not only shock waves but also vortex sheets.

The linearized equations of magneto-gasdynamics for an ideal gas with infinite electrical conductivity in a magnetic field parallel to the velocity have the following form, after the magnetic field is eliminated [1]:

$$
\begin{aligned}
&(1-M^2)\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0 \qquad \left(M=\frac{V_0}{a_0}\right)\\
&[M^2-N^2(1-M^2)]\frac{\partial v_x}{\partial y}-(M^2-N^2)\frac{\partial v_y}{\partial x}=0 \qquad \left(N=\frac{V_a}{a_0}\right)\\
&[M^2-N^2(1-M^2)]\frac{\partial v_x}{\partial z}-(M^2-N^2)\frac{\partial v_z}{\partial x}=0 \qquad \left(V_a=\frac{H_0}{\sqrt{4\pi\rho}}\right)
\end{aligned}
\qquad (1)
$$

Here v_x, v_y and v_z are the perturbation velocities, V_0 the velocity of the unperturbed flow, V_a the Alfven speed, H_0 the unperturbed magnetic field, and ρ_0 the density of the unperturbed flow.

The perturbed magnetic field is connected to the perturbation velocities by the relation

$$\frac{h_x}{H_0} = (1 - M^2)\frac{v_x}{V_0}, \qquad \frac{h_y}{H_0} = \frac{v_y}{V_0}, \qquad \frac{h_z}{H_0} = \frac{v_z}{V_0} \tag{2}$$

Transforming

$$v_x = \frac{M^2 - N^2}{M^2 - N^2(1 - M^2)} v_x^-, \qquad v_y = -v_y^-, \qquad v_z = -v_z^-,$$
$$x = -x^-, \qquad y = y^-, \qquad z = z^- \tag{3}$$

we reduce system (1) into the form

$$\beta^2 \frac{\partial v_x^-}{\partial x^-} - \frac{\partial v_y^-}{\partial y^-} - \frac{\partial v_z^-}{\partial z^-} = 0, \qquad \frac{\partial v_x^-}{\partial y^-} - \frac{\partial v_y^-}{\partial x^-} = 0, \qquad \frac{\partial v_x^-}{\partial z^-} - \frac{\partial v_z^-}{\partial x^-} = 0 \tag{4}$$
$$\beta^2 = \frac{(N^2 - M^2)(1 - M^2)}{M^2 - N^2 \ (1 - M^2)}$$

Evidently β^2 is positive and the equation is hyperbolic for $N/\sqrt{(1 - N^2)} < M < \min(N, 1)$ and for $M > \max(1, N)$.

The first of these regions (quasihyperbolic [1]) represents the more interesting case, where, as mentioned previously, shock waves extend upstream.

The system (4) differs from the corresponding system of linear equations for ordinary gasdynamics only in the absence of the third equation of irrotationality.

From the last two equations of system (2) we obtain

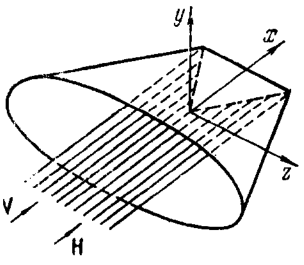
$$\frac{\partial v_y^-}{\partial z^-} - \frac{\partial v_z^-}{\partial y^-} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) = \frac{V_0}{H_0}\left(\frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z}\right) = F(y, z)$$

The function $F(y, z)$ is in general non-zero. However, if at infinity $F(y, z) \equiv 0$, then across weak compression shocks of arbitrary form the function F remains zero. This follows from the fact that the relations satisfied across weak shocks coincide with those for characteristics. By means of transformation (3), these relations reduce to the usual ones of supersonic aerodynamics. Since in ordinary gasdynamics the flow remains irrotational across weak shocks to an accuracy of the cube of the shock strength, the same remains true in the case under consideration, i.e. $F(y, z) \equiv 0$. The function F may differ from zero only in vortex sheets.

Since on the body $v_y^- = -v_y$, $v_z^- = -v_z$, and, moreover, in the subsonic hyperbolic region shock waves leave the side of increasing x^-, then evidently system (4) and boundary conditions describe an ordinary gas flow near the body in the opposite direction, i.e. $V_0^- = -V_0$.

If we take the solution of the corresponding ordinary gas problem with reverse flow to be the solution of the magnetohydrodynamics problem in question, then the vortex sheet generated by the wing in ordinary gas flow will extend upstream in this problem (Figure).

In the supersonic hyperbolic region the flow picture obtained is qualitatively similar to the flow in ordinary aerodynamics. To reduce system (1) to the form (4), it suffices here to replace v_x by v_x^- according to (3), leaving the remaining variables unchanged.



Let us observe some properties of three-dimensional flows, which obtain in both subsonic and supersonic flows.

As shown above, in both cases the solution of the magnetohydrodynamic problem may be reduced to the solution of the corresponding problem in ordinary aerodynamics. Within the limits of linearized theory, the latter solution is irrotational..

However, it is evident that in the resulting flow, the y- and z-components of the vorticity and current will be non-zero. In magnetohydrodynamic shock waves the tangential components of the field and velocity are discontinuous. Consequently, a shock is itself a vortex sheet and a current sheet. In three-dimensional flows the shock strength changes from point to point; hence, the current can flow either into or out of the shock.

On the body surface the velocity and magnetic field undergo a tangential discontinuity. In ordinary aerodynamics the difference in the circulation intensity between two adjacent sections of the wing equals the intensity of the portion of the trailing vortex sheet between the two sections considered. From the solution constructed above it is seen that in the magnetohydrodynamic case the vorticity and current from the boundary layer are not only in the vortex sheet but also throughout the flow, so that the components of the current and vorticity normal to the wing surface are not zero.

Therefore, the currents in the shock, the boundary layer, the vortex sheet and the entire flow form a "single energy system."

We observe that the solution for the quasihyperbolic case constructed above, with a vortex sheet extending upstream, cannot be fully justified within the limits of ideal fluid theory. Nevertheless, it has been established in a series of papers that for sub-Alfven speeds in fluids of infinite conductivity, the viscous wake extends upstream; this fact serves as a justification of the flow picture proposed here, to a certain extent.

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